DELTA HEDGING IN DISCRETE TIME
UNDER STOCHASTIC INTEREST RATE

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Abstract

We examine the effect of stochastic interest rate on the Delta hedging strategy in discrete time when hedging a contingent claim written on a risky asset. The performance of the hedging is mainly measured by the variance of the error. We consider a simple two-dimensional model of the type Black-Scholes combined with the Vasicek model, allowing for correlation between the stock and the interest rate. Within this model, we perform the Delta hedging first by implementing the strategy by taking into account the stochasticity of interest rate and then by using a plain Black-Scholes Delta with deterministic rate. The differences between the two performances can be relevant, mainly depending on the correlation and on the relation between the standard deviation of the risky asset and that of the interest rate. We also consider Delta hedging for an interest rate option in the Cox-Ingersoll and Ross model. The analysis is done by applying a general result for the efficient computation of expected value and variance of the hedging error of a certain class of strategies, which include the Delta strategy.
1 Introduction

One of the most discussed assumptions of financial models, especially criticized in periods of financial turmoils, is that of market completeness, that is the perfect replication of any contingent claim by a suitable dynamic trading strategy. Theoretically, this is often achieved by ruling out any market imperfection, like illiquidity, credit risks, transactions costs, taxes, etc and by assuming the possibility of continuous time trading. Of course, real markets usually fail to satisfy most, if not all of such assumptions. One of the main challenges for financial economics is therefore to address such issues, by proposing models with less stringent hypotheses or by studying what happens when they do not hold. We shall focus on the impossibility of trading continuously in time. Even if all other assumptions of the model are satisfied, the inherent discreteness of trading times is a source of market incompleteness in the real world. The aim of the paper is to efficiently evaluate the impact of trading in discrete time on the final goal of the strategy.

The object of our investigation is the ex-ante assessment of the performances of dynamic trading strategies. Probably, the most notable instance of such problem is measuring the hedging error of a strategy, based on a liquid assets, that tries to hedge a future liability. Problems of such kind arise when replicating either a claim using futures contracts, or a payoff of a derivative security with a delta hedging strategy based on the underlying asset, and in any case when a dynamic strategy is adopted. Ex-ante, a possible way to measure the performance of a strategy is by evaluating expected value and variance of its hedging error. This is usually done by approximations or by Monte Carlo simulations. The approach we propose, based on Laplace transforms, allows to efficiently perform such computations for a very general class of models.

We consider a market model driven by continuous time affine processes, in which by definition the conditional characteristic function is an exponential of an affine function of the state variables (see Duffie et al. (2000) for a formal definition and properties of affine models). In this framework, Angelini and Herzel (2009, 2012) provide semi-closed formulas for the efficient computation of expected value and variance of the hedging error for a quite general class of strategies, called ”affine”, that includes the popular Delta hedging strategy. Such formulas are obtained using a Laplace transform approach, that is based on the idea of writing the payoff of the contingent claim as an inverse Laplace transform. An important feature of the result is that one can study different
type of mispecification. For instance, it is possible to analyse the performance of the standard Black-Scholes Delta strategy when the underlying asset is driven by a process which is not log-normal, like in a stochastic volatility model.

Angelini and Herzel (2012) made the simplifying assumption of deterministic interest rates. In the present work, we extend the analysis to the case of stochastic interest rate. Such extension gives us the opportunity to study the hedging problem in a more general and realistic model. For example, we can study the influence of various interest rate parameters on the hedging performances and also the effect of assuming that the interest rate is deterministic when in fact it is stochastic. For the sake of clearness, we consider a simple two-dimensional affine model, where the underlying evolves according to the Black-Scholes dynamics, while the short-term interest rate follows the process of the Vasicek model, and the stock and the interest rate may be correlated. This is a particular case of a model considered in van Haastrecht et al. (2009) to price long-term derivatives. Notice that if the Cox, Ingersoll and Ross model were used for the interest rate, the resulting two-dimensional model would be affine only in case of zero correlation. Within this model, we implement two types of Delta strategies: the correct model strategy that takes into account the stochasticity of the interest rate, which may be called the model Delta, and the plain Black-Scholes Delta with deterministic rate. We show that the differences between the two strategies may be relevant, mainly depending on the correlation and on the relation between the volatility of the risky asset and that of the interest rate. We conclude that the standard Black-Scholes strategy, still very used by practitioners, may be inappropriate because it may lead to a variance of the error much higher, in relative terms, to that obtained with the correct Delta, especially when the volatility of the interest rates is comparable with that of the stock.

As a final application, we study the Delta hedging for an interest rate option in the Cox, Ingersoll and Ross (1985) model, showing numerical illustrations in the case that the objective measure under which the short rate evolves differs from the risk-neutral measure used to implement the strategy.

2 The general framework

Following Angelini and Herzel (2012), we consider the problem of hedging a European contingent claim with maturity $T$ whose payoff $H$ is represented
as

\[ H = \int_{\mathcal{C}} e^{zyT} \Pi(dz), \tag{2.1} \]

where \( \mathcal{C} = R + i\mathbb{R} \), with \( R \in \mathbb{R} \), \( \Pi \) is a finite complex measure on \( \mathcal{C} \) and \( y_T = \ln(S_T) \), where \( S \) is the price of a risky asset. Examples are the call, the put, the power call and the digital option, see Hubalek et al. (2006).

Angelini and Herzel (2012) consider an affine model in which the interest rate is deterministic. In this paper we extend their results to the case of stochastic interest rate. The log-return \( y = \ln(S) \) of the underlying asset and a short term stochastic interest rate \( r \) are components of a multi-dimensional affine process \( X \), whose other components may include stochastic volatility, dividend yields, etc. The simplest example of such a model is obtained by taking the Black-Scholes dynamics for the underlying and a short rate model for the interest rate, like the Vasicek model. In this case one can allow correlation between stock and interest rate. We will call this model Black-Scholes-Vasicek and it will be examined in section 3.1. If the Cox, Ingersoll and Ross is used for the interest rate, the resulting two-dimensional model would be affine only if the correlation is zero. A model that includes stochastic volatility as well as stochastic interest rate is studied in van Haastrecht et al. (2009). Pan (2002) studied a four-dimensional affine model combining stochastic volatility, interest rates and dividend yield.

More precisely, we let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, P) \) be a filtered probability space satisfying the usual technical conditions. We interpret \( P \) as the physical or objective probability measure. We consider an affine time-homogeneous Markov process \( X \) defined in a state space \( D \subset \mathbb{R}^d \) and we write its conditional characteristic function as

\[ \phi(u, X_t, t, s) = E_t \left[ e^{u^T X_s} \right] = e^{\alpha(u, t, s) + \beta(u, t, s) X_t}, \tag{2.2} \]

where \( u \in i\mathbb{R}^d \), \( t, s \in [0, T] \) with \( t \leq s \), \( E_t \) denotes the expected value conditional on \( \mathcal{F}_t \) and \( \cdot \) the scalar product. The functions \( \alpha(u, t, s) \) and \( \beta(u, t, s) \) go from \( i\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \) to \( \mathbb{C} \) and to \( \mathbb{C}^d \) respectively, and satisfy a system of Riccati equations whose general form is given in Duffie, Pan and Singleton (2000) (Equations (2.5) and (2.6)). We suppose that the functions \( \alpha(u, t, T) \) and \( \beta(u, t, T) \) can be analytically extended to an open convex domain \( U \) containing \( 0 \in \mathbb{C}^d \) for all \( t \in [0, T] \). In this paper we skip technical conditions on the domain \( U \) and we refer to Angelini and Herzel (2012) for details.

We also need the joint characteristic function of \( X \) at times \( t \leq t_1 \leq \)
\[
\cdots \leq t \leq T \text{ conditional on the information in } t,
\]
\[
\phi_{\nu}(u_1, \ldots, u_\nu, X_t, t, t_1, \ldots, t_\nu) = E_t \left[ e^{\sum_{i=1}^\nu u_i X_{t_i}} \right] = e^{\alpha_{\nu}(u_1, \ldots, u_\nu, t, t_1, \ldots, t_\nu) + \beta_{\nu}(u_1, \ldots, u_\nu, t, t_1, \ldots, t_\nu) X_t}
\]

where \(u_k \in i\mathbb{R}^d\) for \(k = 1, \ldots, \nu\). The functions \(\alpha_{\nu}(\cdot)\) and \(\beta_{\nu}(\cdot)\) are equal to \(\alpha(\cdot)\) and \(\beta(\cdot)\) for \(\nu = 1\) and can be computed recursively if \(\nu > 1\) (see Angelini and Herzel (2012) for details).

We also assume that \(X\) is affine under a pricing measure \(Q\). Conditions for a process to be affine under both measures \(P\) and \(Q\) are given by Duffie, Pan and Singleton (2000). We consider the discounted conditional characteristic function
\[
\psi(u, X_t, t, s) = E_t^Q \left[ \exp \left( -\int_t^s r_\tau d\tau \right) e^{u X_s} \right] = e^{\tilde{\alpha}(u, t, s) + \tilde{\beta}(u, t, s) X_t}
\]
The functions \(\tilde{\alpha}(u, t, s)\) and \(\tilde{\beta}(u, t, s)\) solve a system of Riccati equations depending on the risk-neutral dynamics of \(X\). Setting \(u = 0\) in (2.6) we get the discount factor between time \(t\) and \(s\)
\[
P(t, s) = E_t^Q \left[ \exp \left( -\int_t^s r_\tau d\tau \right) \right] = e^{\tilde{\alpha}(0, t, s) + \tilde{\beta}(0, t, s) X_t}
\]
We define the normalized price of the underlying
\[
\bar{S}_t = S_t / P(t, T)
\]
and its increment
\[
\Delta \bar{S}_k = \bar{S}_{t_k} - \bar{S}_{t_{k-1}}
\]
Given a finite and prefixed set of dates from time 0 until maturity \(T\), \(0 = t_0 < t_1 < \ldots < t_N = T\), we let \(\vartheta = (\vartheta_{t_k})\), for \(k = 0, \ldots, N - 1\), be a stochastic process representing a trading strategy. The random variable \(\vartheta_{t_k}\) is the number of shares of \(S\) held from time \(t_k\) up to time \(t_{k+1}\). We assume that it depends only on the information available at time \(t_k\), i.e. that it is
The final value of strategy $\vartheta$ starting from an initial capital $c$ is

$$G_T(\vartheta) = c/P(0,T) + \sum_{k=1}^{N} \vartheta_{t_{k-1}} \Delta \tilde{S}_k \quad (2.7)$$

In (2.7) we are assuming that all portfolio readjustments are invested in or borrowed from the money market account. The hedging error of the strategy is then given by

$$\varepsilon(\vartheta, c) = H - G_T(\vartheta). \quad (2.8)$$

As in Angelini and Herzel (2012) (Definition 3.1), we consider strategies of the following affine form:

$$\vartheta_{t_k} = \int c e^{a(z,t_k) + b(z,t_k) \cdot X_{t_k}} \Pi(dz), \quad (2.9)$$

for all $k = 0, \ldots, N - 1$, where $a(z, t_k)$ and $b(z, t_k)$ are functions from $\mathbb{C} \times \mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively. We skip technical conditions on the functions $a(z, t_k)$ and $b(z, t_k)$, referring again to Angelini and Herzel (2012) for details.

From (2.6) and using (the conditional version of) Fubini’s Theorem, we get an expression for the value at time $t$ of a European claim with payoff expressed as in (2.1)

$$H_t = E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) H \right] = E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) \int_c e^{z y_T} \Pi(dz) \right] = \int_c E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) e^{z 1_y X_T} \right] \Pi(dz) = \int_c e^{\bar{a}(z 1_y, t, T) + \bar{b}(z 1_y, t, T) \cdot X_t} \Pi(dz) \quad (2.10)$$

where $1_y$ denotes the $d$-dimensional vector of zeros except for the entry corresponding to $y$ that is equal to one. By differentiating (2.10), we can compute the sensitivities of the pricing formula with respect to the factors of the model. The Delta of the claim is shown in Angelini and Herzel (2012). If we differentiate with respect to the interest rate, we obtain the Rho of the claim.
at time $t$

$$\Delta^H_t = \frac{\partial H_t}{\partial r_t} = \frac{\partial}{\partial r_t} \int_{\mathcal{C}} e^{\bar{a}(z_{1_y,t},T) + \bar{\beta}(z_{1_y,t},T) \cdot X_t} \Pi(dz)$$

$$= \int_{\mathcal{C}} \bar{\beta}(z_{1_y}, t, T) \cdot 1_r e^{\bar{a}(z_{1_y},t,T) + \bar{\beta}(z_{1_y},t,T) \cdot X_t} \Pi(dz) \quad (2.11)$$

where $1_r$ denotes the $d$-dimensional vector of zeros except for the entry corresponding to $r$ that is equal to one. Hence, the Rho strategy is an example of strategy of the form (2.9) with

$$a(z, t) = \ln(\bar{\beta}(z_{1_y}, t, T) \cdot 1_r) + \bar{\alpha}(z_{1_y}, t, T)$$

$$b(z, t) = \bar{\beta}(z_{1_y}, t, T)$$

The hedging error (2.8) of a strategy of the form as in (2.9) for a contingent claim whose payoff can be written as (2.1), has the following integral representation

$$\varepsilon(\vartheta, c) = -c/P(0, T) + \int_{\mathcal{C}} \left( e^{\varphi r} - \sum_{k=1}^{N} e^{a(z,t,k-1)+b(z,t,k-1) \cdot X_{t,k-1} \Delta S_k} \right) \Pi(dz). \quad (2.12)$$

We have the following result, which is a generalization of the main result in Angelini and Herzel (2012) to the case of stochastic interest rates.

**Theorem 2.1** Let $H$ be a contingent claim satisfying condition (2.1), $\vartheta$ be strategy of the form as in (2.9), and $c$ be the initial capital, then

$$E[\varepsilon(\vartheta, c)] = \int_{\mathcal{C}} e(z) \Pi(dz) - c/P(0, T) \quad (2.13)$$

and

$$E[\varepsilon(\vartheta, 0)^2] = \int_{\mathcal{C}} \int_{\mathcal{C}} (v_1(w, z) - v_2(w, z) - v_3(w, z) + v_4(w, z)) \Pi(dw) \Pi(dz) \quad (2.14)$$
The expressions for $e(z)$, $v_i(w,z)$, $i = 1, 2, 3, 4$, are analogous to those of Theorem 3.1 in Angelini and Herzel (2012) and are relegated to the Appendix, together with the proof of the theorem. Theorem 2.1 states that the expected value and the variance of the hedging error may be represented respectively as a one-dimensional and a two-dimensional inverse Laplace transforms. It may be used to study the effects of model misspecification or trader personal views, in terms of hedging strategies and parameters, on the performance of the hedge. This because the claim, the model and the strategy are completely independent from each other. As example, in Section 3.1, we will consider the use of a plain Black-Scholes Delta, implemented considering a deterministic interest rate, in the two-dimensional model Black-Scholes-Vasicek where instead the interest rate is stochastic. This is an example of misspecified strategy; it will be compared with the correct model Delta, which takes into account the stochasticity of interest rate.

Formulas (2.13) and (2.14) can be evaluated numerically through numerical inversion of one-dimensional and two-dimensional Laplace transform. For more details on this as well as numerical integration schemes and tests we refer to Angelini and Herzel (2009) and Angelini and Herzel (2012). In Section 3.2, we provide a test for the algorithm by comparing the results obtained via a Monte Carlo simulation in the Cox, Ingersoll and Ross model.

3 Numerical illustrations

3.1 The Black-Scholes-Vasicek model

In this model the state variable $X = (y, r)$ has two components, $y = \ln(S)$ and $r$ the stochastic interest rate. They are driven by the following dynamics, which we write under the risk-neutral measure.

$$dy_t = (r_t - \frac{1}{2}\sigma_y^2)dt + \sigma_y dW^1_t \tag{3.15}$$

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dW^2_t \tag{3.16}$$

We suppose that the two Brownian motions are correlated with correlation coefficient $\rho$. This is a combination of the Black-Scholes model with the Vasicek model for the short term interest rate. In the degenerate case where $\sigma_r = 0$, the short rate process is deterministic, and if in addition $\theta = r_0$, one recovers the Black-Scholes model with constant rate.
This model is affine and one can write the Riccati Equations (2.5) and (2.6) in Duffie, Pan and Singleton (2000) for $\alpha(u,t,T)$ and $\beta(u,t,T) = (\beta_1(u,t,T), \beta_2(u,t,T))$, with $u = (u_1, u_2)$, that satisfy

$$E_t \left[ e^{-\eta \int_t^T \frac{r_s}{r_T} ds} e^{u_1 y_T + u_2 r_T} \right] = e^{\alpha(u,t,T) + \beta_1(u,t,T)y_t + \beta_2(u,t,T)r_t},$$

where $\eta$ could be either 0 or 1. For simplicity, in our numerical analysis we suppose that the objective measure and the risk-neutral measure coincide, hence $\alpha(u,t,T)$ and $\beta(u,t,T)$ above are those involved both in Equation (2.6) and, when $\eta = 0$, in Equation (2.2). The differential equations for $\beta_1$ and $\beta_2$ are particularly simple to solve. The function $\alpha$ is determined by a tedious but straightforward integration. We do not report the results here because they are a particular case of those obtained in van Haastrecht et al. (2009). Notice that if we replaced the Vasicek dynamics for the short rate model with the CIR dynamics, the model would be affine if and only if the correlation between stock and interest rate is zero, hence the model would be much less flexible, if analytic tractability were required.

In this model, there are two interesting hedging strategies: the first is the model Delta, where in Equation (2.11) we use the model’s $\bar{\alpha} = \alpha$ and $\bar{\beta} = \beta$, the second is the standard Black-Scholes Delta, for which $\bar{\alpha}$ and $\bar{\beta}$ are given according to that model and the drift would be given by a deterministic risk-free rate. Of course in the model, the risk-free rate will change with time and the trader has to insert a value for it at each rebalancing date $t_k$. To do so, he or she will naturally extrapolate it from the price of a riskless bond as

$$\bar{r}_{t_k} = -\frac{\log(P(t_k,T))}{T-t_k} = -\frac{\bar{\alpha}(0,t_k,T) + \bar{\beta}(0,t_k,T)r_{t_k}}{T-t_k},$$

This is an example of mispecified strategy, as it neglects the stochasticity of interest rates. Since it is a common strategy used in practice, it is of interest to compare its hedging performance with that of the first strategy.

We consider a European call option written on the risky asset $S$ with maturity $T = 0.5$ years and strike $K = S_0 = 100$. For our analysis, we fix once for all some of the parameter of the model: the initial risk-free rate $r_0 = 0.05$, and the drift parameters $\theta = r_0$, $\kappa = 0.05$ and the volatility of the underlying $\sigma_y = 0.3$. We also fix the number of rebalancing dates to be $N = 12$ (roughly twice a month), but the results are analogous for different values of $N$, the only difference being the level of the variance of hedging errors, which obviously decreases with $N$. We are interested in analyzing
the effect on the variance of the hedging error of the correlation coefficient $\rho$ that we will let vary in the set $[-0.8, -0.6, -0.3, 0, 0.3, 0.6, 0.8]$ and of the relation between the volatility $\sigma_y$ and that of the interest rate $\sigma_r$, hence we let $\sigma_r$ assume the values $[0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5]$. In particular, we will compare the variances of the hedging error of the two Delta strategies described above. In Figures 1 and 2 we represent our results. Figure 1 shows the increasing effect of the correlation $\rho$ on the variance of the hedging error for both strategies. When the volatility of the interest rate is small $\sigma_r = 0.01$ (top panel) the two strategies perform in a similar matter, while for higher volatilities, respectively $\sigma_r = 0.15$ and 0.3 (middle and bottom panel), the two strongly differ. In relative terms, the variance of the hedging error of the Black-Scholes Delta is higher than that of the model Delta going from less than 1\% for $\sigma_r = 0.01$ to 10-16\% for $\sigma_r = 0.15$ up to 27-65\% for $\sigma_r = 0.3$. The range of values for each $\sigma_r$ is due to the different correlation coefficients and it is higher for values near 0, as it is clear from the figure. In Figure 2 we illustrate, for three different values of $\rho$, -0.6, 0 and 0.6, the impact on the variance of the interest rate volatility compared to that of the underlying. On the x-axis we indeed represent the ratio $\sigma_r/\sigma_y$. Notice that, for $\rho = -0.6$ and in general for negative correlations, the variance of the model Delta decreases with $\sigma_r$, while for zero and positive values increases.

### 3.2 Affine Short Rate Models

We show how to apply Theorem 2.1 to the computation of the Delta strategy in the case of affine short rate models, in particular we write the integral representation of a contingent claim written on a zero coupon bond and the related Delta hedging strategy. In this case the process $X$ is the one-dimensional process of the short rate. The functions $\alpha(u, t, T)$ and $\beta(u, t, T)$ in (2.2) and $\bar{\alpha}(u, t, T)$ and $\bar{\beta}(u, t, T)$ in (2.6) may be computed explicitly in some important cases as the models by Cox, Ingersoll and Ross (1985) or Vasicek (1977). We consider here the case of Cox, Ingersoll and Ross model. The dynamics of the short rate are given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma \sqrt{\tau} dW_t$$ (3.17)

We think of the dynamics (3.17) as under the objective measure $P$ and we write the market price of risk as $q = -\pi/\sigma \sqrt{\tau}$ to get the dynamics under a martingale measure $Q$, so that the drift of (3.17) under $Q$ is $\kappa \theta - (\kappa - \pi)r_t$. 


Figure 1 Black-Scholes-Vasicek model. Variances of model Delta and Black-Scholes Delta hedging strategies for a European call option with maturity $T_1 = 0.5$ years written on the risky asset as a function of the correlation $\rho$ for $\sigma_r = 0.01$ (Top), $\sigma_r = 0.15$ (middle) and $\sigma_r = 0.3$ (bottom). The volatility of the underlying is $\sigma_y = 0.3$ and the number of rebalancing dates is $N = 12$. 
Figure 2  Black-Scholes-Vasicek model. Variances of model Delta and Black-Scholes Delta hedging strategies for a European call option with maturity $T_1 = 0.5$ years written on the risky asset as a function of the ratio $\frac{\sigma_r}{\sigma_y}$ for $\rho = -0.6$ (Top), $\rho = 0$ (middle) and $\rho = 0.6$ (bottom). The volatility of the underlying is $\sigma_y = 0.3$ and the number of rebalancing dates is $N = 12$. 
For our purposes, we look for affine exponents \( \alpha(u, t, T) \) and \( \beta(u, t, T) \) such that

\[
E_t \left[ e^{-\eta \int_t^T r_s \, ds} e^{ur_T} \right] = e^{\alpha(u, t, T) + \beta(u, t, T)r_t}
\]

for \( u \in \mathbb{C} \). If \( \eta = 0 \), we think of this as Equation (2.2) with expectation taken under the objective measure \( P \). If \( \eta = 1 \), using the dynamics under \( Q \), we think of this as Equation (2.6) for \( \bar{\alpha}(u, t, T) \) and \( \bar{\beta}(u, t, T) \). The explicit expressions for the Vasicek and for the Cox, Ingersoll and Ross models can be found in Filipović (2009), Section 10.3.2.1, 10.3.2.2. We report the case of Cox, Ingersoll and Ross model for convenience of the reader. The Riccati equations in this case are, for \( \tau = T - t \), (see Filipović (2009), Theorem 10.4, or Duffie, Pan and Singleton (2000), Equations (2.5) and (2.6))

\[
\begin{align*}
\dot{\alpha}(\tau) &= \kappa \theta \beta(\tau) \\
\dot{\beta}(\tau) &= \frac{1}{2} \sigma^2 \beta^2(\tau)^2 - (\kappa - \pi) \beta(\tau) - \eta
\end{align*}
\]

with boundary conditions \( \alpha(0) = 0, \beta(0) = u \). Setting

\[
\begin{align*}
d &= \sqrt{(\kappa - \pi)^2 + 2\eta \sigma^2} \\
a &= \frac{\kappa - \pi + d}{\sigma^2} \\
b &= \frac{\kappa - \pi - d}{\sigma^2}
\end{align*}
\]

the solutions are

\[
\begin{align*}
\beta(\tau) &= \frac{b(a - u) - a(b - u)e^{-d\tau}}{(a - u) - (b - u)e^{-d\tau}} \\
\alpha(\tau) &= \frac{\kappa \theta}{\sigma^2} \left( -2 \log \left( \frac{(a - u) - (b - u)e^{-d\tau}}{a - b} \right) + ((\kappa - \pi) - d)\tau \right)
\end{align*}
\]

Let us now consider a European claim \( H \) maturing at date \( T_1 \), written on the zero coupon bond with maturity \( T_2 > T_1 \). Hence

\[
S_t = P(t, T_2) = e^{\tilde{\alpha}(0, t; T_2) + \tilde{\beta}(0, t; T_2)r_t}
\]

and \( y_t = \ln(S_t) = \tilde{\alpha}(0, t; T_2) + \tilde{\beta}(0, t; T_2)r_t \) is an affine function of \( X_t = (r_t) \). The statement of Theorem 2.1 in this case has to be slightly modified.
Formula (2.1) may be written as

\[
H = \int_C e^{\gamma t_1} \Pi(dz) = \int_C e^{\tilde{a}(0,T_1,T_2)z + \tilde{\beta}(0,T_1,T_2)z r t_1} \Pi(dz).
\]

Hence, the price of the claim at time \( t \) can be computed as

\[
H_t = E_t^Q \left[ \exp \left( - \int_t^{T_1} r_s ds \right) \int_C e^{\gamma t_1} \Pi(dz) \right] = \int_C e^{\tilde{a}(0,T_1,T_2)z} \psi(\tilde{\beta}(0,T_1,T_2)z, r_t, t, T_1) \Pi(dz) = \int_C e^{\tilde{a}(0,T_1,T_2)z} e^{\tilde{a}(\tilde{\beta}(0,T_1,T_2)z, t, T_1)} + \tilde{\beta}(0,T_1,T_2)z r_t \Pi(dz).
\]

Therefore the derivative of \( H_t \) with respect to the factor \( r_t \) is

\[
D_t = \frac{\partial H_t}{\partial r_t} = \int_C \frac{\partial}{\partial r_t} e^{\tilde{a}(0,T_1,T_2)z} e^{\tilde{a}(\tilde{\beta}(0,T_1,T_2)z, t, T_1)} + \tilde{\beta}(0,T_1,T_2)z r_t \Pi(dz) = \int_C \tilde{\beta}(\tilde{\beta}(0,T_1,T_2)z, t, T_1) e^{\tilde{a}(0,T_1,T_2)z + \tilde{a}(\tilde{\beta}(0,T_1,T_2)z, t, T_1)} + \tilde{\beta}(0,T_1,T_2)z r_t \Pi(dz).
\]

and the Delta of the claim is

\[
\Delta_t = \frac{\partial H_t}{\partial S_t} = D_t \frac{1}{\tilde{\beta}(0, t, T_2) S_t}
\]

Both \( \Delta \) and \( D \) are strategies which have an affine representation, in particular the Delta has

\[
a(z, t) = \ln \left( \frac{\tilde{\beta}(\tilde{\beta}(0, T_1, T_2)z, t, T_1)}{\tilde{\beta}(0, t, T_2)} \right) + \tilde{\alpha}(0, T_1, T_2)z + \tilde{\alpha}(\tilde{\beta}(0, T_1, T_2)z, t, T_1) - \tilde{\alpha}(0, t, T_2)
\]

\[
b(z, t) = \tilde{\beta}(\tilde{\beta}(0, T_1, T_2)z, t, T_1) - \tilde{\beta}(0, t, T_2).
\]
For the numerical illustration, we take parameters estimated in Duan and Simonato (1999), namely $\kappa = 0.1644$, $\theta = 0.0648$, $\sigma = 0.0438$ and we set $r_0 = 0.06$. We set the risk premium $\pi = 0$, so that the objective measure and the pricing measure are the same, analogously to the Heston’s model case. We consider an at-the-money forward European option with maturity $T_1 = 1$ written on a bond with maturity $T_2 = 10$ with notional of $10^3$. Since we also wish to assess the validity of our algorithm with a Monte Carlo simulation, we computed a 95% confidence band for the simulated value of the standard deviation of the hedging error. We remark that in this case we can implement an exact simulation. Figure 4 shows the standard deviation of the Delta hedging error as the number of trading intervals increases together with the confidence interval.

4 Appendix

Theorem 2.1 Let $H$ be a contingent claim satisfying condition (2.1), $\vartheta$ be strategy of the form as in (2.9), and $c$ be the initial capital, then

$$ E[\varepsilon(\vartheta, c)] = \int_C e(z)\Pi(dz) - c/P(0, T) \quad (4.18) $$

and

$$ E[\varepsilon(\vartheta, 0)^2] = \int_C \int_C (v_1(w, z) - v_2(w, z) - v_3(w, z) + v_4(w, z))\Pi(dw)\Pi(dz) \quad (4.19) $$

where

$$ e(z) = \phi(z1_y, X_0, 0, T) - \sum_{k=1}^N e^{a(z, t_{k-1})} \times $$

$$ (e^{-\bar{\alpha}(0, t_k, T)} \phi_2(b(z, t_{k-1}), -\bar{\beta}(0, t_k, T) + 1_y, X_0, 0, t_{k-1}, t_k) - $$

$$ e^{-\bar{\alpha}(0, t_k-1, T)} \phi_2(b(z, t_{k-1}), -\bar{\beta}(0, t_{k-1}, T) + 1_y, X_0, 0, t_{k-1}) $$

$$ v_1(w, z) = \phi((w + z)1_y, X_0, 0, T) $$

$$ v_2(w, z) = \sum_{k=1}^N e^{a(w, t_{k-1})} \times $$

$$ (e^{-\bar{\alpha}(0, t_k, T)} \phi_3(b(w, t_{k-1}), -\bar{\beta}(0, t_k, T) + 1_y, z1_y, X_0, 0, t_{k-1}, t_k, T) - $$

$$ e^{-\bar{\alpha}(0, t_k-1, T)} \phi_3(b(w, t_{k-1}), -\bar{\beta}(0, t_{k-1}, T) + 1_y, z1_y, X_0, 0, t_{k-1}, T) $$

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Figure 3 Variances (top), standard deviation over the price of the option (middle), expected value over the price of the option (bottom) of Delta hedging for a European call option with maturity $T_1 = 1$ year written on a zero coupon bond with maturity $T_2 = 2$ years as a function of the number of trading intervals $N$. CIR model with parameters $r_0 = 0.06$, $\kappa = 0.1644$, $\theta = 0.0648$, $\sigma = 0.0438$ and $\pi = [-0.1237, 0, 0.1237]$. 

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Figure 4 Standard deviation of Delta hedging error for European call option with maturity $T_1 = 1$ year written on a zero coupon bond with maturity $T_2 = 10$ years and notional $10^3$, as a function of the number of trading intervals $N$. CIR model with parameters $r_0 = 0.06$, $\kappa = 0.1644$, $\theta = 0.0648$, $\sigma = 0.0438$ and $\pi = 0$. The results of the algorithm are compared with a 95% confidence band obtained by simulations.
\[ v_2(w, z) = v_2(z, w). \]

To write \( v_4(w, z) \), we define

\[
\bar{m}(j_1, j_2, k_1, k_2) =
\begin{cases}
\phi_4(b(w, t_{j_1}), -\bar{\beta}(0, t_{j_2}, T) + 1_y, b(z, t_{k_1}), -\bar{\beta}(0, t_{k_2}, T) + 1_y, X_0, 0, t_{j_1}, t_{j_2}, t_{k_1}, t_{k_2}) \\
\phi_4((b(z, t_{k_1}), -\bar{\beta}(0, t_{k_2}, T) + 1_y, b(w, t_{j_1}), -\bar{\beta}(0, t_{j_2}, T) + 1_y, X_0, 0, t_{k_1}, t_{k_2}, t_{j_1}, t_{j_2})
\end{cases}
\]

for \( j_1 \leq j_2 \leq k_1 \leq k_2 \);

Then we have

\[
v_4(w, z) = \sum_{j=1}^{N} \sum_{k=1}^{N} e^{a(w, t_{j-1})} e^{a(z, t_{k-1})} \times
\]

\[
(e^{-\bar{a}(0, t_{j-1}, T)} - \bar{a}(0, t_{k-1}, T)) \bar{m}(j - 1, j, k, k - 1, k - 1)
- e^{-\bar{a}(0, t_{j-1}, T)} - \bar{a}(0, t_{k-1}, T)) \bar{m}(j - 1, j, k - 1, k - 1)
- e^{-\bar{a}(0, t_{j-1}, T)} - \bar{a}(0, t_{k-1}, T)) \bar{m}(j - 1, j, k, k - 1, k - 1)
+ e^{-\bar{a}(0, t_{j-1}, T)} - \bar{a}(0, t_{k-1}, T)) \bar{m}(j - 1, j, k - 1, k - 1, k - 1)).
\]

Therefore, the variance of the hedging error is

\[
\text{var}(\varepsilon(\vartheta, c)) = \text{var}(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2.
\]

Proof. The proof is along the same lines as that of Theorem 3.1 in Angelini and Herzel (2012) to which we refer for technical details. Given
(2.12), applying Fubini’s Theorem we have,

\[
E \left[ H - \sum_{k=1}^{N} \vartheta_{t_k-1} \Delta \bar{S}_k \right] = \\
= \int_C \left\{ E \left[ e^{zyT} \right] - \sum_{k=1}^{N} E \left[ e^{a(z,t_k-1) + b(z,t_k-1) \cdot X_{t_k-1}} \Delta \bar{S}_k \right] \right\} \Pi(dz) = \\
= \int_C \left\{ E \left[ e^{zyT} \right] - \sum_{k=1}^{N} e^{a(z,t_k-1)} \times \\
E \left[ e^{b(z,t_k-1) \cdot X_{t_k-1}} \left( e^{\alpha(0,t_k,T) - \beta(0,t_k,T) X_{t_k}} - e^{\mu_{t_k-1} - \alpha(0,t_k-1,T) - \beta(0,t_k-1,T) X_{t_k-1}} \right) \right] \right\} \Pi(dz) = \\
= \int_C \left\{ E \left[ e^{zyT} \right] - \sum_{k=1}^{N} e^{a(z,t_k-1)} \times \\
\left( e^{-\alpha(0,t_k,T)} \cdot E \left[ e^{b(z,t_k-1) \cdot X_{t_k-1} + (-\beta(0,t_k,T) + 1) \cdot X_{t_k-1}} \right] \right) \right\} \Pi(dz) = \\
= \int_C \left\{ \phi(z, 1_y, X_0, 0, T) - \sum_{k=1}^{N} e^{a(z,t_k-1)} \times \\
\left( e^{-\alpha(0,t_k,T)} \cdot \phi_2 \left( b(z, t_k-1), -\beta(0, t_k, T) + 1_y, X_0, 0, t_k-1, t_k \right) - \\
e^{-\alpha(0,t_k-1,T)} \cdot \phi \left( b(z, t_k-1) - \beta(0, t_k-1, T) + 1_y, X_0, 0, t_k-1 \right) \right) \right\} \Pi(dz)
\]
which is (4.18). To prove (4.19) we need to compute
\[
E \left[ (H - \sum_{k=1}^{N} \vartheta_{t_k} \Delta \bar{S}_k)^2 \right] = \\
= E \left[ \int_{\mathcal{C}} \left( e^{z y T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \Pi(dz) \right] \\
= E \left[ \int_{\mathcal{C}} \left( e^{w y T} - \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \Pi(dw) \right] \\
= E \left[ \int_{\mathcal{C}} \int_{\mathcal{C}} E \left[ \left( e^{z y T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \times \left( e^{w y T} - \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \Pi(dz) \Pi(dw) \right] \right].
\]

We can again apply Fubini’s Theorem, so we get
\[
E \left[ (H - \sum_{k=1}^{N} \vartheta_{t_k} \Delta \bar{S}_k)^2 \right] = \\
= \int_{\mathcal{C}} \int_{\mathcal{C}} E \left[ \left( e^{z y T} - \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \times \left( e^{w y T} - \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1}) \cdot X_{t_k-1} \Delta \bar{S}_k} \right) \right] \Pi(dz) \Pi(dw).
\]

Let us compute all the expectations needed:
\[
E \left[ e^{(z+w) y T} \right] = \phi((z+w)1_y, X_0, 0, T).
\]
\[
E \left[ e^{zyt} \sum_{k=1}^{N} e^{a(w,t_{k-1})+b(w,t_{k-1}) \cdot X_{t_{k-1}}} \Delta \tilde{S}_k \right] = \\
= \sum_{k=1}^{N} e^{a(w,t_{k-1})} E \left[ e^{b(w,t_{k-1}) \cdot X_{t_{k-1}}} e^{zyt} \Delta \tilde{S}_k \right] = \\
= \sum_{k=1}^{N} e^{a(w,t_{k-1})} \times \\
\left( e^{-\bar{\alpha}(0,t_k)} \sum_{j=1}^{N} e^{a(w,t_{j-1}) \cdot X_{t_{j-1}}} \bar{\beta}(0,t_k,T) \cdot X_{t_k} + y_{t_k} + zyT \right) \\
= \sum_{k=1}^{N} e^{a(w,t_{k-1})} \times \\
\left( e^{-\bar{\alpha}(0,t_k)} \sum_{j=1}^{N} e^{a(w,t_{j-1}) + (-\bar{\beta}(0,t_k,T) + 1y) \cdot X_{t_k} + z1y(X_{t_k})} \right) \\
= \sum_{k=1}^{N} e^{a(w,t_{k-1})} \times \\
\left( e^{-\bar{\alpha}(0,t_k)} \phi_2(b(w,t_{k-1}), -\bar{\beta}(0,t_k,T) + 1y, z1y, X_{0,0}, t_{k-1}, t_k, T) - e^{-\bar{\alpha}(0,t_k)} \phi_2(b(w,t_{k-1}) - \bar{\beta}(0,t_{k-1}, T) + 1y, z1y, X_{0,0}, t_{k-1}, T) \right) \\
= \psi_2(w, z).
\]

The expectation
\[
E[ e^{zyt} \sum_{k=1}^{N} e^{a(z,t_{k-1})+b(z,t_{k-1}) \cdot X_{t_{k-1}}} \Delta \tilde{S}_k]
\]
is obtained as above after interchanging \( w \) with \( z \).

The last term is
\[
E \left[ \sum_{j=1}^{N} \sum_{k=1}^{N} e^{a(w,t_{j-1})+b(w,t_{j-1}) \cdot X_{t_{j-1}}} \Delta \tilde{S}_j e^{a(z,t_{k-1})+b(z,t_{k-1}) \cdot X_{t_{k-1}}} \Delta \tilde{S}_k \right] = \\
= \sum_{j=1}^{N} \sum_{k=1}^{N} e^{a(w,t_{j-1})} e^{a(z,t_{k-1})} E \left[ e^{b(w,t_{j-1}) \cdot X_{t_{j-1}}} e^{b(z,t_{k-1}) \cdot X_{t_{k-1}}} \Delta \tilde{S}_j \Delta \tilde{S}_k \right].
\]
Expanding the products \( \Delta \tilde{S}_j \Delta \tilde{S}_k \)

one gets \( v_4(w, z) \).

\( \square \)

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