ON AN IMPLICIT ASSESSMENT OF FUZZY VOLATILITY IN THE BLACK AND SCHOLES ENVIRONMENT

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On an implicit assessment of fuzzy volatility in the Black and Scholes environment

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Abstract

In this work we suggest a methodology to obtain the membership of a non observable parameter through implicit information. To this aim we profit from the interpretation of membership functions as coherent conditional probabilities. We develop full details for the well known Black and Scholes pricing model where the membership of the volatility parameter is obtained from a sample of either asset prices or market prices for options written on that asset.

Keywords: Fuzzy membership elicitation, Implicit Information, Coherent Conditional Probability Assessments and Extension, Probability-Possibility Transformation

1 Motivation and assumptions

Most mathematical models rely on unknown quantities (parameters) which are usually estimated through sampling techniques. One of the main issues in the application of such models is the additional problem of non-observability of some of these parameters. In this case direct sampling is not possible and it is compulsory to rely on indirect information. A typical example in financial applications is the derivation of the volatility of a risky asset. In particular, in the context of Black and Scholes model [4] (BS model, hereafter), it is
assumed that the price \( S_t \) of the risky asset follows a geometric Brownian motion:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,
\]  

(1)

where \( \sigma \) is the volatility of the asset, \( t \) is the time index, \( \mu \) the mean asset return and \( W_t \) a standard Wiener process. Under this dynamics assumption, a closed formula can be derived for the evaluation of European-style derivatives, e.g. the price at time \( t \) for a European Call option with strike price \( K \) and expiration time \( T \) is:

\[
C(S_t) = \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)},
\]  

(2)

with

\[
d_{1,2} = \frac{\ln(S_t/K) + (r \pm \sigma^2(T-t))}{\sigma \sqrt{T-t}},
\]  

(3)

\( r \) the continuously compounded riskless interest rate, and \( \Phi(\cdot) \) the cdf of the standard normal distribution.

Note that the pricing formula (2) only depends on the rate \( r \) and on the volatility parameter \( \sigma \). Since \( r \) is indeed independent of the stock price, \( \sigma \) is the most relevant parameter in (1). As a consequence, its estimation is crucial and has inspired a huge literature on estimation procedures (see, among others, [3],[30],[33]). Besides, generalizations of BS model have been introduced in which \( \sigma \) is assumed to be random; seminal papers in this direction are [26, 34, 39] and [24].

However, even in the simple constant case, the estimation of volatility is debatable. In fact, while volatility is not directly observable, indirect information is available through associated quantities such as the price \( S_t \) of the risky asset itself as well as the price of some of its derivative claims. Pure statisticians usually base their evaluations on a sample of stock prices and estimate \( \sigma \) by the standard deviation of sample log-returns.

On the contrary, market practitioners mostly rely on indirect information inferred from a sample of market option prices, which they claim to be more informative about the current market beliefs than asset prices themselves. There is anyhow a general agreement that both kinds of available information can only produce vague statements about the value of \( \sigma \).

Recently, several authors proposed to explicitly include imprecision in parameters estimation by grading their admissibility through membership
functions. Within the BS environment, in [40] it is assumed that the stock price at time $t$ as well as the parameters $r$ and $\sigma$ are triangular fuzzy-numbers. This assumption is relaxed in [23] where other shapes for the memberships of the fuzzy values are adopted. By applying different methodologies both contributions derive a fuzzy price for Call and Put options with crisp maturities and strike prices. We stress that the support, the core and the shape of fuzzy variables in the applications suggested therein are assumed as known (or preliminarily assigned). In [9] the authors apply BS model to price real options; they assume both the stock price and the strike of the option to be modelled as trapezoidal fuzzy numbers $\tilde{S}_t$ and $\tilde{K}$. The Call option price is obtained as a fuzzy number by applying a hybrid version of formula (2), where the value in (3) is computed by replacing $S_t$ and $K$ with the possibilistic mean value of their fuzzified version, while the volatility parameter $\sigma$ is replaced by the square-root of the possibilistic variance of the fuzzy stock price $\tilde{S}_t$. Again, the support and the shape of fuzzy values $\tilde{S}_t$ and $\tilde{K}$ are initially assigned. A different approach is adopted in [36] where the authors assume that volatility is described through a fuzzy number $\tilde{\sigma}$. They make use of the Heston stochastic volatility model [24], where the probability density function for the instantaneous variance is explicitly given. The authors profit from this density function and from a probability/possibility transformation in order to obtain the membership function for the fuzzy volatility $\tilde{\sigma}$. Once again, in their example, parameters for the known distribution are given values. Similar remarks apply to other fuzzy generalizations of the BS environment (see, among others, [25] and [41]).

In this paper, we agree on introducing vagueness-imprecision for the volatility parameter in BS model; our contribution is to propose a methodology for a proper elicitation of $\tilde{\sigma}$ which goes further the pre-assignment adopted in the quoted papers. This is done by using indirect information obtained from either asset or derivatives prices. In addition, contrarily to what commonly done in literature, the volatility parameter and its estimate are treated as two distinct entities. Nevertheless, we are able to infer on $\sigma$ by observing the behavior of each selected estimator $\hat{\theta}$ through an intermediate pseudo-membership which permits the flow of information from the estimator $\hat{\theta}$ to the parameter $\sigma$.

In order to elicit such pseudo-memberships we profit from the interpretation of membership functions as coherent conditional probabilities provided in [11, 14, 15] and from their possible extensions. In fact, coherent conditional probability can be looked at as a general non-additive "uncertainty
measure” \( \mu(\cdot) = P(E|\cdot) \) of the conditioning events. This gives rise to a clear, precise and mathematical frame, which allows to define fuzzy subsets and to bound memberships of other fuzzy sets. Moreover, this interpretation permits the use of Bayesian methodologies to get a proper membership function by joining different pieces of probabilistic information.

Of course, our approach could be straightforwardly extended to any other field where a parametric model is involved and where the evaluation of an unknown and unobservable parameter is affected both from the available sources of indirect information and from sample variability. Potential applications could be in engineering (e.g. [6]) or in models for climate changes (see, among others, [29]).

The paper is organized as follows: in Section 2 we describe the theoretical framework where the membership elicitation procedure can be embedded: in Subsec. 2.1 we briefly recall the main notions of coherence while in Subsec. 2.2 we formalize the procedure to obtain the searched membership \( \mu_\tilde{\sigma}(x) \) through scenarios identification, prior and likelihoods elicitation, pseudo-membership determination and the eventual probability/possibility transformation. In Section 3 we describe the two sources of information to be considered in the numerical application which is fully developed in the subsequent Section 4. Section 5 concludes and gives some hints for future developments.

2 Obtaining the membership via Bayes Rule or coherent extension

Let us sketch the aforementioned interpretation of memberships as coherent conditional assessments [11, 14, 15]. The arguments usually brought forward to distinguish grades of membership from probabilities often refer to a restrictive interpretation of event and probability. On the contrary we interpret probability as a measure of belief in given propositions. In particular a conditional probability \( P(E|H) \) can be directly introduced through coherence without resorting to the usual ratio between unconditional probabilities \( P(E \land H) \) and \( P(H) \), ratio which needs \( P(H) \) to be positive.

Going into details, the formal interpretation of fuzzy subset given in [14] through coherent conditional probability is:

**Definition 1** Given a random quantity \( X \) with range \( C_X \) and a related property \( \varphi \), a fuzzy subset \( E_\varphi^* \) of \( C_X \) is the pair \((E_\varphi, \mu_{E_\varphi})\) with
• $E_\varphi$ is the proposition “An assessor claims $\varphi$”.

• $\mu_{E_\varphi}(x) = P(E_\varphi | X = x)$ is a coherent conditional probability, looked on as a real function of the argument $x \in C_X$.

A coherent conditional probability $P(E_\varphi | X = x)$ is, hence, a measure of how much an assessor, given the event $(X = x)$, is willing to claim the property $\varphi$, and it plays the role of the membership function of the fuzzy subset $E_\varphi^*$. To let the paper be self-contained, we briefly introduce the basic concepts of the conditional coherence paradigm.

2.1 Coherence

Coherence for partial conditional probabilities can be reduced to the property of compatibility with the well established mathematical model of the so called full conditional probabilities, as introduced by [18] and in line also with [16, 28, 32]. Full conditional probabilities are characterized by the following set of axioms:

**Definition 2** Given a Boolean algebra $\mathcal{B}$, a full conditional probability on $\mathcal{B} \times \mathcal{H}^0$ (with $\mathcal{H} \subseteq \mathcal{B}$ closed with respect to logical sums, and putting $\mathcal{H}^0 = \mathcal{H} \setminus \{\emptyset\}$) is a function $P : \mathcal{B} \times \mathcal{H}^0 \to [0,1]$ such that

(i) $P(\cdot| H)$ is a finitely additive probability on $\mathcal{B}$ for any given $H$ in $\mathcal{H}^0$;

(ii) $P(H|H) = 1$ for all $H \in \mathcal{H}^0$;

(iii) $P(A \land B|C) = P(B|C)P(A|B \land C)$ for every $A, B \in \mathcal{B}, C, B \land C \in \mathcal{H}^0$, where $\land$ denotes the usual logical conjunction.

In discrete settings, the Boolean algebra $\mathcal{B}$ is usually the power set $\mathcal{P}(\Omega)$ of some sample space $\Omega$. What is peculiarly relevant is the possibility to assess and handle conditional probabilities given on domains without any particular structure. In fact we have:

**Definition 3** If $\mathcal{E} = [A_1|H_1,\ldots,A_n|H_n]$ is an arbitrary set of conditional events, an assessment $P(\cdot|\cdot)$ on $\mathcal{E}$ is said to be coherent if there exists a full conditional probability $P'(\cdot|\cdot)$ defined on $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$ (with $\mathcal{P}(\Omega)$ the power set of sample space $\Omega$ spanned by the events $A_1, H_1,\ldots,A_n, H_n$) which coincides with $P(\cdot|\cdot)$ on $\mathcal{E}$. 
Whereas, for infinite domains $\mathcal{E}$, coherence is equivalent to coherence of $P(\cdot|\cdot)$ on any finite subset $\mathcal{F} \subset \mathcal{E}$ ([13, §11.3]).

Contrary to usual approaches, within this view conditional probabilities can be given directly without the use of joint and marginal evaluations. An operational check of coherence is anyway possible thanks to the following characterization theorem ([10, 11, 12, 13]):

**Theorem 4** Let $\mathcal{E}$ be an arbitrary finite family of conditional events and $\Omega$ denote the set of atoms $\omega_r$ generated by the events $A_1, H_1, \ldots, A_n, H_n$. For a real function $P$ on $\mathcal{E}$ the following two statements are equivalent:

(a) $P$ is a coherent conditional probability assessment on $\mathcal{E}$;

(b) there exists (at least) a class of probabilities $\{P_0, P_1, \ldots, P_k\}$, each probability $P_\alpha$ being defined on a suitable subset $\Omega_\alpha \subseteq \Omega$, such that for any $A_i|H_i \in \mathcal{E}$ there is a unique $P_\alpha$ with

\[
\sum_{\omega_r|H_i} P_\alpha(\omega_r) > 0 \quad P(A_i|H_i) = \frac{\sum_{\omega_r|A_i\land H_i} P_\alpha(\omega_r)}{\sum_{\omega_r|H_i} P_\alpha(\omega_r)};
\]

moreover, $\Omega_\alpha' \subset \Omega_\alpha''$ for $\alpha' > \alpha''$ and $P_{\alpha''}(\omega_r) = 0$ if $\omega_r \in \Omega_\alpha'$.

Any class $\{P_0, P_1, \ldots, P_k\}$ singled-out by condition (b) is said to agree with the conditional probability $P$.

With regard to our purposes, what is crucial is the possibility to make inference from a coherent assessment to any new target conditional event. In fact the following theorem, essentially due to [16], holds:

**Theorem 5** Let $P(\cdot|\cdot)$ be an assessment on an arbitrary family $\mathcal{E}$; then there exists a (possibly not unique) coherent extension of $P$ to any family $\mathcal{K} \supset \mathcal{E}$ if and only if $P(\cdot|\cdot)$ is coherent on $\mathcal{E}$.

Thanks to the characterization given by the aforementioned Theorem 4, it is possible to obtain such a coherent extension through specific linear optimization problems based on the classes $\{P_0, P_1, \ldots, P_k\}$ (for details refer e.g. to [8, 11]).

The role of coherence is simply that of allowing not structured domain. A subsequent step is to identify an effective method for its evaluation, as done in what follows for our purposes.
2.2 Coherent membership elicitation

Going back to our original problem, we have to look at an operational way to assess a membership function for the volatility parameter $\sigma$. To follow closely Def.1, we deal with some vague statement about $\sigma$, e.g. “$\sigma$ is around a specific value”, and we have to grade the willingness of an assessor to claim it, conditionally to the actual, but unknown, value of the parameter. More specifically, we should look for

$$\mu_{H_s}(x) = P(\sigma \text{ “is claimed to be a value around ” } \bar{\sigma}_s | \sigma = x),$$

(4)

where different values of $\bar{\sigma}_s$ are representatives of distinct market behavior scenarios $H_s$, $s = 1, \ldots, n_s$, while $x$ is the precise value attained by the parameter $\sigma$. Given the non-observability of the parameter, beliefs about $\sigma$ must be based on some estimator so what we can actually elicit is

$$\tilde{\mu}_{H_s}(\hat{\theta}) = P(H_s | Info^{\hat{\theta}}),$$

(5)

where $Info^{\hat{\theta}}$ is any information gathered from an estimator $\hat{\theta}$ of $\sigma$. We profit from pseudo-memberships (5) to select the most plausible scenario(s) for the volatility parameter $\sigma$, which will serve the final elicitation of membership (4) as detailed afterwards. As already remarked, the volatility parameter and its estimate are seen as two distinct entities. However, by observing the behavior of $\hat{\theta}$, we are able to infer on the true parameter $\sigma$.

Let us assume, for computational purposes, that any empirical or simulated distribution for $\hat{\theta}$ is discretized into $n_b$ classes $\hat{\theta}_b$, named "bins". With a little abuse we set $\hat{\theta}_b \equiv (\hat{\theta} \in \hat{\theta}_b)$ so that values for the pseudo-memberships (5) are computed "bin by bin" with $Info^{\hat{\theta}} \equiv \hat{\theta}_b$. By using the notation introduced in Subsec. 2.1, the input domain $E$ is given by

$$\{ H_s, \hat{\theta}_b | H_s \}_{s=1, \ldots, n_s; b=1, \ldots, n_b}$$

i.e. by the unconditional scenarios ranges and by the realization of the estimator inside one of its possible bins, conditioned to different market scenarios.

As a first step we need the pseudo-memberships (5); they can be derived either via Bayes rule, whenever scenarios $H_s$, $s = 1, \ldots, n_s$, come from an exhaustive partition, or via coherent extension, whenever scenarios come from partial knowledge, i.e. they overlap, or they do not cover all the possibilities or there is some logical constraint among them and possible values of the parameter. In any case, the computation of (5) is based on likelihoods $P(\hat{\theta}_b | H_s)$ and priors $P(H_s)$. In the numerical application we obtain
the priors by experts evaluations about scenarios based on historical data, whereas likelihoods are derived through simulation. More precisely, we assume that, by randomly generating values of $\sigma$ from a specific distribution $\pi_s$ for each scenario $H_s$, we are able to obtain the distribution for $\hat{\theta}$ conditioned to $H_s, s = 1, 2, ..., n_s$. The latter step relies on the availability of an explicit relation between $\sigma$ and $\hat{\theta}$. Obviously, different approaches may lead to different assessments.

Once likelihoods $P(\hat{\theta}_b|H_s)$ are obtained, it is possible to infer on the probabilities $P(\hat{\theta}_b)$. If we have full information, i.e. the scenarios form a partition and there is not any logical constraint among the $H_s$ and the $\hat{\theta}_b$, then the available assessment $P(\cdot|\cdot)$ on $E$ is surely coherent [37, Prop. 1]. In this case we obtain the $P(\hat{\theta}_b)$ through the usual disintegration formula

$$P(\hat{\theta}_b) = \sum_{s=1}^{n_s} P(\hat{\theta}_b|H_s)P(H_s),$$  

and the pseudo-membership $P(H_s|\hat{\theta}_b)$ by Bayes rule

$$P(H_s|\hat{\theta}_b) = \frac{P(\hat{\theta}_b|H_s)P(H_s)}{P(\hat{\theta}_b)}. \quad (7)$$

Otherwise, whenever information is partial, overall conditional coherence of the assessment $P(\cdot|\cdot)$ on $E$ must be checked and, once it has been ensured, we can extend it by the procedures detailed in [8, 11] to $H_s|\hat{\theta}_b$, obtaining coherent intervals

$$[P_*(H_s|\hat{\theta}_b), P^*(H_s|\hat{\theta}_b)]. \quad (8)$$

Given the incompatibility of the various bins $\hat{\theta}_b$, Theorem 2 in [14] guarantees the coherence of any value inside the intervals (8). Hence, we obtain a set of plausible pseudo-memberships instead of a single one, so that we have to deal with interval type-2 [22] pseudo-memberships (see e.g. Fig. 1). The further step is to consider the current (observed) value $\hat{\theta}_{obs}$ of the parameter estimator. On the base of the bin $\hat{\theta}_b$ including $\hat{\theta}_{obs}$ we can select most plausible scenarios $H_s$ by maximizing $P(H_s|\hat{\theta}_b)$. Tails among type-1 or interval incomparability among type-2 pseudo-memberships can induce more than one plausible scenario; in this case, any of such scenarios can be a valid candidate so we may take the disjunction of them. Note that we assume the set of pseudo-memberships $\tilde{\mu}_{H_s}(\hat{\theta}), s = 1, 2, ..., n_s$ to hold for a suitable time period, hence its computation is performed once in a while.
At this point it is possible to elicit the searched membership \( \mu_\sigma(x) \) by transforming the simulating distributions \( \pi_\pi \) associated with the selected scenario(s). At first, membership \( \mu_{H_\pi}(x) \) is obtained through a probability/possibility transformation [20] of the simulating distribution for each selected scenario. In particular, for the sake of computational simplicity, we adopt the following:

\[
\mu_{H_\pi}(x) = \begin{cases} 
2F_{\pi_\pi}(x) & \text{if } x \leq \sigma_{\pi_\pi}^M \\
2(1 - F_{\pi_\pi}(x)) & \text{if } x > \sigma_{\pi_\pi}^M 
\end{cases}
\]

where \( F_{\pi_\pi}(x) \) and \( \sigma_{\pi_\pi}^M \) denote, respectively, the cumulative distribution function and the median of the simulating distribution \( \pi_\pi \) associated to \( H_\pi \). Note that such probability-possibility transformation is, among those proposed in [21], that one induced by confidence intervals around the median.

In case a single most plausible scenario \( H_\pi \) is selected, we obtain a fuzzy number and \( \mu_\sigma(x) \equiv \mu_{H_\pi}(x) \), otherwise, \( \mu_\sigma(x) \) may be obtained through a reasonable \( t \)-conorm of the memberships associated to each selected scenario.

\section{Indirect information: the choice of input data}

If we believe in the efficiency of financial markets, then the price of an asset represents the agents opinion, i.e. the total information available on that asset. Stock and derivative assets are traded in different financial markets
which provide different sources of indirect information on the volatility of the asset. As already mentioned, we can infer the volatility $\sigma$ as the sample standard deviation of asset returns; in this case indirect information is based on a sample of stock prices. Alternatively, we can obtain the volatility as implied by the price of some derivative assets written on that stock e.g., given the price of a traded option, we may use the so-called BS implied volatility, obtained by numerical inversion of the BS option pricing formula (2) with respect to the volatility parameter $\sigma$. As well, we may profit, when available, from the value of a volatility index, usually obtained by considering as input a set of suitably selected traded Call and Put options. The two latter procedures both rely on a sample of derivatives prices.

In our numerical application we aim at the elicitation of a membership function for the volatility of the S&P500 Index. Hence, $S_t$ is the value at time $t$ of the Index and as alternative estimators of $\sigma$ we consider either the sample standard deviation of the Index log-returns or the VIX volatility index, as defined in what follows. The former is based on a time series of the S&P500 returns while the latter is based on a sample of Call and Put options prices.

### 3.1 Historical data of the underlying

In the first approach we estimate the volatility $\sigma$ by means of the historical volatility, which is computed, given a time series $x_i, i = 1, 2, ..., n$ of the S&P500 Index returns, as:

$$
\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}},
$$

(10)

where $\bar{x}$ is the sample mean. In order to elicit the priors on scenarios $H_s, s = 1, 2, ..., n_s$ we use the empirical distribution of $\hat{\sigma}$ obtained by the largest available time series. More precisely, we compute $\hat{\sigma}$ on $m$ (overlapping) moving windows with $n$ observations each. As remarked in Subsec. 2.2, the empirical distribution of $\hat{\sigma}$ is discretized into $n_b$ classes $\hat{\sigma}_b$ and represented through the associated histogram (see Fig.2 (a)). On the base of such empirical distribution, an expert can choose the number $n_s$ of scenarios and their relative ranges. The easiest way to do it is on the base of specific quantiles for $\hat{\sigma}$. Then, for each scenario $H_s$, we count the number $m_s$ of

values falling in that scenario and we estimate $P(H_s)$ as

$$P(H_s) = \frac{m_s}{m}.$$  

We stress that if $H_s, s = 1, 2, \ldots, n_s$ is a partition then $\sum_{s=1}^{n_s} m_s = m$ so that $\sum_{s=1}^{n_s} P(H_s) = 1$.

Furthermore, once scenarios $H_s, s = 1, 2, \ldots, n_s$ are determined, we compute the associated likelihoods for $\hat{\sigma}|H_s$ as follows:

- we randomly generate $N$ values for the volatility in the scenario $H_s$ from a known distribution $\pi_s$, namely $\sigma_i, i = 1, 2, \ldots, N$;
- for each of these values $\sigma_i$, we simulate, through the BS model (1), a sample $x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}$ of $n$ values for the Index return;
- for each sample of returns we then compute the corresponding value of $\hat{\sigma}(i)$, for $i = 1, 2, \ldots, N$, as

$$\hat{\sigma}(i) = \sqrt{\frac{\sum_{t=1}^{n}(x_t^{(i)} - \bar{x}^{(i)})^2}{n - 1}}.$$  

Note that the $N$ values for $\hat{\sigma}(i)$ are obtained from distribution $\pi_s$ conditionally on the scenario $H_s$ selected at the beginning of the procedure; the dependence on $s$ has been omitted to simplify the notation. Such $N$ values are also summarized in $n_b$ classes and the likelihoods discretized into histogram densities $P(\hat{\sigma}_b|H_s)$, $b = 1, \ldots, n_b$.  

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Once such likelihoods are obtained, we can infer on pseudo-memberships \( P(H_s | \tilde{\sigma}_b) \) as in (7) or at least compute their coherent bounds (8). On the base of the actual value of the estimator \( \tilde{\sigma}_{obs} \), it is possible, by comparison, to find the most plausible scenarios and obtain the searched memberships \( \mu_{\tilde{\sigma}}(x) \) as described in Subsec. 2.2.

3.2 Market option prices and the VIX

The alternative setting for the volatility estimator relies on a time series for the VIX, which is a volatility index obtained through a set of prices for options written on the S&P500 Index (SPX options). Let us define

\[
V_T^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2,
\]

(11)

where \( T \) is the time to maturity, \( F \) is the forward index level (derived from index option prices), \( K_0 \) is the first strike below \( F \), \( K_i \) is the strike price of the \( i \)-th out-of-the-money option and \( \Delta K_i \) is the interval between strikes. \( R \) is the risk free rate while \( Q(K_i) \) is the midpoint of the bid-ask values for option with strike \( K_i \).

The selected options are out-of-the-money SPX Calls and Puts centered around an at-the-money strike price \( K_0 \). Only SPX options quoted with non-zero bid prices are used in the VIX calculation. The squared VIX value is finally obtained as a weighted mean of \( V_{T_1}^2 \) and \( V_{T_2}^2 \) with \( T_1 \) and \( T_2 \) corresponding to near and to next term maturities. For more details on VIX computation see [38].

Note that the VIX is expressed as a percentage; we adopt \( \nu = \text{VIX}/100 \) as a proxy for the volatility (i.e. \( \widehat{\theta} = \nu \)) so that the distribution of interest is simply given by the empirical distribution of \( \nu \).

Assume we are given a time series of length \( m \) for \( \nu \); similarly to the previous subsection, we can discretize its empirical distribution into \( n_b \) bins (see Fig.2 (b)). On the base of this distribution experts can grasp \( n_s \) relevant scenarios and consequently elicit the priors as \( P(H_s) = \frac{m_s}{m} \). Afterwards, likelihoods are obtained as below:

- we randomly generate \( N \) values for the volatility in the scenario \( H_s \) from a known distribution \( \pi_s \), namely \( \sigma_i, i = 1, 2, ..., N \);
• for each simulated $\sigma_i$, we compute, according to BS option pricing formula, the Call and Put option prices for the two maturities and for the selected Strikes needed for the calculation of the VIX;

• we compute the corresponding value of the VIX index to get $\nu(i)$, for $i = 1, 2, ..., N$.

• the proportion of $\nu(i)$ falling in each of the $n_b$ classes can be adopted as a discrete density approximation $P(\nu_b|H_s)$.

Once again, through the current value $\nu_{obs}$, we can obtain the searched membership $\mu_{\tilde{\sigma}}(x)$ by the probability/possibility transformation of the simulating functions associated to the scenarios $H_s$ with highest pseudo-membership support $P(H_s|\nu_{obs})$ (see Subsec. 2.2).

4 Numerical Illustration

In this section we describe how to elicit the membership function for the volatility parameter in practice. We apply either $\tilde{\sigma}$ or $\nu$, introduced in the previous section, and we consider the largest time series available for the S&P500 Index as well as for the VIX Index; the former starts on January 3, 1950, the latter on January 2, 1990 and both end on October 10, 2010\(^1\). The (discretized) distributions of $\tilde{\sigma}$ and $\nu$, from which we derive priors $P(H_s)$, are shown in Fig.2 and compared in Fig.3.

Figure 3: Prior distribution comparison for $\tilde{\sigma}$ (solid line) and $\nu$ (dashed line) estimators.

\(^1\)Data are downloaded from the Yahoo Finance web-site.
We take into account three or five volatility scenarios; namely $n_s = 3$ corresponds to scenarios of low, medium and high volatility and $n_s = 5$ to scenarios of low, medium-low, medium, medium-high and high volatility. These scenarios are associated to intervals of the real line, which may be disjoint as well as overlapping, obtained by dividing the range of the corresponding estimator according to its percentiles.

Three different parametric functions $\pi_s$ are chosen to simulate the volatility in order to obtain the conditional likelihoods. At first, the Uniform distribution is adopted as representative of minimal knowledge; as an alternative the Log-normal distribution is considered since, as pointed out in [35], it resembles the density function of the integrated volatility in the Heston model (see [24]). Furthermore, the Gamma distribution is selected to obtain the simulated variance since it describes well the invariant distribution of the local variance of the stock in the Garch diffusion setting (see [1]). As prototypes, we fully describe here three cases as reported in Table 1, where $q_a$ represents the $a$-th percentile of $\hat{\theta}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$n_s$</th>
<th>L</th>
<th>ML</th>
<th>M</th>
<th>MH</th>
<th>H</th>
<th>Simul.D.</th>
</tr>
</thead>
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<td>1</td>
<td>3</td>
<td>$[q_0, q_{25}]$</td>
<td>-</td>
<td>$[q_{25}, q_{75}]$</td>
<td>-</td>
<td>$[q_{75}, q_{100}]$</td>
<td>U</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$[q_0, q_{30}]$</td>
<td>$[q_{20}, q_{80}]$</td>
<td>-</td>
<td>$[q_{70}, q_{100}]$</td>
<td>LN</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>$[q_0, q_{20}]$</td>
<td>$[q_{20}, q_{40}]$</td>
<td>$[q_{40}, q_{60}]$</td>
<td>$[q_{60}, q_{80}]$</td>
<td>$[q_{80}, q_{100}]$</td>
<td>G</td>
</tr>
</tbody>
</table>

Table 1: Several scenarios detection based on $\hat{\theta}$ estimator’s quantiles, scenario labels (L=low, ML=medium-low, M=medium, MH=medium-high, H=high) and simulating distributions (U=uniform, LN=log-normal, G=gamma).

It is possible to analyze each case study following the proposed methodology.

In our exercise the VIX time series provides itself the $m = 5244$ observations for $\nu$ while we obtain a sample of $m = 15254$ values for $\hat{\sigma}$ by considering moving windows of length $n = 45$ from the time series of the S&P500 log-returns. Concerning the derivation of likelihoods, $N$ values for the parameter $\sigma$ are generated through the density function $\pi_s$ (Uniform, Log-normal or Gamma) with specific parameters for each scenario $H_s$; in Case 1 $N = m$ while in Case 2 and Case 3 $N = 100000$ to guarantee observations in the tails. The number of bins is fixed at $n_b = 50$ for all the distributions, while the effective ranges will depend on the estimator at hand.
It is worth noticing that, although the parameter $\sigma$ is obtained by simulation from $\pi_s$, the corresponding estimators are derived as described in Subsecs. 3.1, 3.2 and depend on the implicit relation between the parameter and its estimates. The distributions of the estimators are different from the original distributions $\pi_s$ for $\sigma$. For example in Case 1, even starting from disjoint uniform $\pi_s$, we get empirical distributions of $\hat{\sigma}$ and $\nu$ which overlaps and are mostly not uniform (see Fig.4).

Figure 4: Empirical distributions of $\hat{\sigma}$ (a) and $\nu$ (b) for Case 1.

In Cases 1 and 3 it is possible to apply plain Bayes rule (7) to obtain the pseudo-memberships (Figs. 5-6), since the scenarios are partitions.

Figure 5: pseudo-memberships $P(H_s|\hat{\theta})$, $s = L, M, H$ for case 1 with $\hat{\theta} \equiv \hat{\sigma}$ (a) and $\hat{\theta} \equiv \nu$ (b).

On the other hand, in Case 2 the scenarios overlap, hence the best we can obtain is the interval type-2 pseudo-memberships as plotted in Fig. 7. Let us assume that the current values for the estimators are $\hat{\sigma}_{obs} = 0.16$ and $\nu_{obs} = 0.19$, highlighted with a vertical dotted line in all the aforementioned
The most plausible scenarios are uniquely determined for Cases 1 and 3, both for \( \hat{\sigma} \) and \( \nu \). In Case 1 we select the "medium" (\( s = M \)) scenario and, by the probability-possibility transformation (9) of the Uniform simulating distribution \( \pi_M \), we get the memberships for \( \sigma \) to be "around" 0.124 and "around" 0.1923, respectively (see Fig.8). Similarly, in Case 3 we get the "medium-high" (\( s = MH \)) scenario and, by the probability-possibility transformation (9) of the associated distribution \( \pi_{MH} \), we get \( \sigma \) "around" 0.147 and 0.23, respectively (see Fig.9).

Note that in Case 1 the current value \( \hat{\sigma}_{obs} \) is not included in the support of the fuzzy number elicited for \( \sigma \); similarly, in Case 3 we get a negligible membership value for \( \nu_{obs} \). If, on one side, this could result counterintuitive, on the other hand it emphasizes the difference between the real parameter \( \sigma \) and the estimated values.
and its estimators $\hat{\sigma}$ and $\hat{\nu}$. In fact, the elicited memberships concern possible values for $\sigma$ based on the observed values $\hat{\sigma}_{obs}$ and $\hat{\nu}_{obs}$ of the estimators which are two representatives of indirect information.

For Case 2, and in particular for $\hat{\sigma}_{obs} = 0.16$, we get two overlapping plausibility intervals. This produces two most plausible scenarios to be selected so the more general approach is to take both the ”medium” and the ”high” scenarios as supported by the current observation. Hence, we transform the two simulating distributions into two different plausible memberships which are respectively “around” 0.1182 and “around” 0.2024 (see Fig.10(a)). Several choices are possible, among known $t$-conorms of the two memberships (see, e.g. [19]), to get the final membership for $\sigma$. Differently, on the base of the VIX estimator we select a single scenario and get an evaluation of $\sigma$ to be “around” 0.1980 (see Fig.10(b)).

5 Conclusive remarks and future directions

In this work we give a contribution to research in stochastic models with fuzzy parameters. The novelty of our approach is to avoid a pre-assignment of memberships and to suggest a methodology to elicit them.

To this aim we profit from the interpretation of membership functions as coherent conditional probabilities and from their possible extensions. With this approach we embed the elicitation procedure into a rigorous mathematical frame which also allows to join different pieces of probabilistic information.
by applying generalized Bayesian methodologies.

In our application we focus on the elicitation of a proper membership function $\mu_{\tilde{\sigma}}(x)$ for the volatility parameter in Black and Scholes model. Moreover, since volatility is not-observable, we base our elicitation procedure on indirect information obtained either from a sample of asset prices or from a set of observed derivatives values.

A peculiarity of our contribution is that the volatility parameter and its estimates are treated distinctly. Though, by observing the behavior of each selected estimator, we are able to infer on the original volatility parameter, by introducing an intermediate pseudo-membership which permits the flow of information from one to the other. Our approach could be extended to other parametric models as well as to different application fields.

The proposed numerical exercise shows the feasibility of the method and emphasizes the fact that different sources of information can be adopted. Indirect information obtained respectively through the historical volatility or through the VIX volatility Index lead to different conclusions. To jointly profit from them, an aggregation or merging of the two is compulsory. Specifically, there are two stages where this integration might be performed: at the pseudo-memberships level ($P(H_s|\tilde{\sigma})$ and $P(H_s|\nu)$) or at the final memberships one ($\mu_{\sigma|\tilde{\sigma}_{obs}}(s)$ and $\mu_{\sigma|\nu_{obs}}(s)$). Within the former choice we might adopt a specific connective depending on the semantic properties one wants to represent (see e.g. [5, 19]) or, on the contrary, to apply a more general approach by considering all possible coherent extensions of the two (as depicted in [14]). On the other side, working directly with the final memberships, we...
Figure 10: Fuzzy numbers for $\sigma$ deriving from medium or high volatility scenarios associated to $\sigma_{obs} = 0.16$ (a) and medium scenario associated to $\nu_{obs} = 0.19$ (b) and with Log-Normal simulated likelihood.

could average the two by computing a fuzzy mean (see e.g. [17]) or merge them by some specific pseudo-distance minimization (see e.g. [7]). Of course, appropriateness of such a choice will depend on practical consequences for the specific application. Since in this paper we do not face this issue, we do not develop any further analysis on this aspect that anyhow will deserve our attention in the next future.

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